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HW10 , Math 531, Spring 2014

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QUESTION 1. (i) Let $p \geq 3$ be a prime integer. Prove that $x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1 \in Z[x]$ is an irreducible polynomial.

(ii) Find a monic polynomial $d(X) \in Q[X]$ such that $Q[X]/(d(X))$ is ring-isomorphic to $K = Q(\sqrt{2} + \sqrt{3})$. What is $[Q(\sqrt{2} + \sqrt{3}) : Q]$? [Hint : Let $X = \sqrt{2} + \sqrt{3}$. Then find $d(X)$ by working it backward!, so $X^2 = 5 + 2\sqrt{6}$, hence $(X^2 - 5) = 2\sqrt{6}$ blabla.....]. Find a basis for $Q(\sqrt{2} + \sqrt{3})$ over Q.

(iii) Let F be a field, $d(X) \in F[X]$ of degree 3. Prove that $d(X)$ is irreducible over F if and only if $d(X)$ has no roots in F.

(iv) Prove that $x^3 + 9x^2 + 5x + 1$ is irreducible over Q.

(v) Let $F \subset K$ be field extensions and $\alpha \in K$ be an algebraic number over F. Prove that every element in $F(\alpha)$ is an algebraic number. [Hint: Let $n = [F(\alpha) : F]$ and let $a \in F(\alpha)$. We need to show that $f(a) = 0$ for some $f(x) \in F[x]$. Consider the set $S = \{1, a, ..., a^n\}$. If the elements in S are not distinct, then find such f(x)!!! (easy!!). If the elements in S are distinct, then they are dependent since |S| > n. So ...it is easy now to find your f(x)]

(vi) Let $F \subset K$ be field extensions and assume that $\alpha \in K$ be an algebraic number over F. Prove that α^{-1} is an algebraic number over F and $F(\alpha) = F(\alpha^{-1})$. Assume that $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in F[x]$ is irreducible and $f(\alpha) = 0$. Let $b = a_0^{-1}$. Prove that $g(x) = x^n + ba_1x^{n-1} + ba_2x^{n-2} + ba_3x^{n-3} + ... + ba_{n-1}x + b$ is irreducible over F. [Hint: What is $g(\alpha^{-1})$?]

(vii) Let R be the field of all real numbers, and let $f(x)$ be a non-constant irreducible polynomial over R. Prove that $deg(f) = 1$ or 2. [Hint: $R[i] = C$ and $[C : R] = ...$]

(viii) Let $F = Q(\sqrt[4]{3})$ and $K = F(\sqrt[6]{3})$. Find the unique monic irreducible polynomial, say $f(x)$, over F such that $f(\sqrt[6]{3}) = 0$. Find a basis for K over F.

(ix) Let $F \subset K$ be field extensions and assume that $a, b \in K$ be algebraic numbers over F. Assume that $f(x)$ is an irreducible polynomial over F such that $f(a) = f(b) = 0$. Prove that $F(a)$ is ring-isomorphic to $F(b)$.

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